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On Some Spherical t-Designs

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Let G be a finite subgroup of the orthogonal group $O(d)$. It is shown that many spherical t-designs are constructed from G, if some particular irreducible representations of $O(d)$ remain irreducible when restricted to G.

Let Ω_d be the unit sphere in the Euclidean space $\mathbb{R}^d$ of dimension d. A subset X of Ω_d is said to be a spherical t-design if the cardinality of X is finite and if

$$\sum_{\xi \in X} f(\xi) = 0$$

for all homogeneous harmonic polynomials f of degree $1, 2,..., t$ (see Delsarte, Goethals and Seidel [3].) Many examples of spherical t-designs have been given in [3] together with important properties of spherical designs.

Our purpose in the present paper is to add some further examples of spherical t-designs, which are constructed from certain finite subgroups in the orthogonal group $O(d)$.

THEOREM 1. *Let G be a finite subgroup of $O(d)$, and let the natural representation of G be irreducible. Then for any element $\mathbf{X}$ in Ω_d the subset $X = \{g\mathbf{X} \mid g \in G\}$ in Ω_d is a spherical 2-design.*

More generally, we have

THEOREM 2. *Let ρ_l be the irreducible representation of degree d_l of the orthogonal group $O(d)$ on the space of the lth spherical functions on Ω_d, where $d_l = \binom{d+l-1}{d-1} - \binom{d+l-3}{d-1}$.* ($\rho_l$ appears as an irreducible component of the l *th* (but not the ith ($i \leqslant l-1$)) symmetrized Kronecker product of the natural representation of $O(d)$.)

(i) *Suppose that G is a finite subgroup in $O(d)$ and that the restriction $\rho_l \mid G$ of the representation ρ_l to G is irreducible for $l = 0, 1,..., s$. Then for any $\mathbf{X} \in \Omega_d$ the subset $X = \{g\mathbf{X} \mid g \in G\}$ in Ω_d is a spherical $2s$-design.*

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(ii) *In addition to the assumption in* (i) *suppose that the restriction* $\rho_{\sigma+1} \mid G$ *does not contain any* $\rho_l \mid G$ $(l = 0, 1,..., s)$ *as irreducible component. Then for any* $\mathbf{X} \in \Omega_d$ *the subset* $X = \{g\mathbf{X} \mid g \in G\}$ *in* Ω_d *is a spherical* $(2s + 1)$-*design.*

Proof of Theorem 1. (This is nothing but a special case of the proof of Theorem 2 which is given later. However, this wil indicate the basic idea of the present paper very clearly.) Since $O(d)$ is transitive on Ω_d, we may assume without loss of geneality that $\mathbf{X} = {}^t(1, 0,..., 0)$. Let $a_{ij}(g)$ be the (i, j)-entry of the matrix $g \in G$ $(\subset O(d))$. Then $g\mathbf{X} = {}^t(a_{11}(g), a_{21}(g),..., a_{d1}(g))$. The space of homogeneous harmonic polynomials of degree 1 is generated by the monomials x_i $(i = 1, 2,..., d)$, and the space of homogeneous harmonic polynomials of degree 2 is generated by the x_ix_j $(i \neq j)$ and the $x_i^2 - x_j^2$ $(i \neq j)$. Now, from the orthoogonality relations among the entries of irreducible representations (cf. [2, Formula (31.5)]), we obtain that

$$\sum_{g \in G} a_{i1}(g) = 0, \tag{1}$$

$$\sum_{g \in G} a_{i1}(g)\, a_{j1}(g) = \sum_{g \in G} a_{i1}(g)\, a_{1j}(g^{-1}) = 0, \tag{2}$$

and

$$\begin{aligned}\sum_{g \in G} (a_{i1}(g)\, a_{i1}(g) - a_{j1}(g)\, a_{j1}(g)) &= \sum_{g \in G} (a_{i1}(g)\, a_{1i}(g^{-1}) - a_{j1}(g)\, a_{1j}(g^{-1})) \\ &= (| G |/d) - (| G |/d) \\ &= 0. \end{aligned} \tag{3}$$

Thus, we have the assertion of Theorem 1 immediately.

Proof of Theorem 2. First, let us recall some fundamental properties of the representation theory of $O(d)$ and the theory of spherical functions on Ω_d (see, for example, [10, 6, 8, 9, 1, 5] and others for the details.) Let ρ_l $(l = 0, 1, 2,...)$ be the lth spherical representation of $O(d)$ of degree d_l. Then the representation ρ_l can be regarded as an orthogonal representation on the space V_l of all homogeneous harmonic polynomials of degree l. Let $\langle\ ,\ \rangle$ be the G-invariant inner product on V_l. As an orthonormal base of V_l, we can take

$$\{u_{l_1,l_2,...,l_d} \mid l_1,..., l_d \in \mathbf{Z},\ l = l_1 \geqslant l_2 \geqslant \cdots \geqslant l_{d-1} \geqslant | l_d |\}.$$

For $l_1,..., l_d \in \mathbf{Z}$ with $l = l_1 \geqslant l_2 \geqslant \cdots \geqslant l_{d-1} \geqslant | l_d |$, let the functions $\varphi_{l_1,l_2,...,l_d}(x)(x \in O(d))$ be defined by

$$\varphi_{l_1,l_2,...,l_d}(x) = \langle u_{l_1,l_2,...,l_d}, \rho_l(x)\, u_{l_1,0,...,0}\rangle.$$

Then the functions $\varphi_{l_1,l_2,\ldots,l_d}(x)$ are functions of x_{11}, x_{21},..., x_{d1} if we write

$$x = (x_{ij}(x)) = (x_{ij}) \in O(d),$$

and are called spherical functions on the sphere

$$\Omega_d = \{(x_{11},\ldots, x_{d1}) \mid x_{11}^2 + \cdots + x_{d1}^2 = 1\} \ (= O(d)/O(d-1)).$$

Moreover, the functions $\varphi_{l_1,l_2,\ldots,l_d}(x_{11}, x_{21},\ldots, x_{dl})$ are homogeneous harmonic polynomials in x_{11}, x_{21},..., x_{dl} of degree l_1. Now, let $d\Omega$ denote the surface element in Ω_d, then we get

$$\int_{\Omega_d} \varphi_{l_1,l_2,\ldots,l_d}(x) \cdot \varphi_{l_1',l_2',\ldots,l_d'}(x)\, d\Omega = \begin{cases} 1/d_{l_1}, & \text{if } l_1 = l_1',\ l_2 = l_2',\ldots,\ l_d = l_d', \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Now, we can easily show that $d_{l+1} > d_l$ (for $l = 0, 1, 2,\ldots$) for $d \geqslant 3$. Thus, for $d \geqslant 3$ if $\rho_l \mid G$ are irreducible for $l = 0, 1,\ldots, s$, then the $\rho_l \mid G$ ($l = 0, 1,\ldots, s$) are mutually distinct. This last assertion is directly verified also for $d = 2$ without difficulty, because we may assume that G is irreducible and so is a dihedral group. Since the functions $\varphi_{l_1,l_2,\ldots,l_d}(g)$ $(g \in G)$ give certain entries of the matrices of the representation $\rho_{l_1} \mid G$, from the orthogonality relations of the entries of irreducible representations (see [2, (31.5)]), we obtain that

$$\frac{1}{|G|} \sum_{g\in G} \varphi_{l_1,l_2,\ldots,l_d}(g) \cdot \varphi_{l_1',l_2',\ldots,l_d'}(g) = \begin{cases} 1/d_{l_1} & \text{if } l_1 = l_1',\ l_2 = l_2',\ldots,\ l_d = l_d', \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

provided that $l_1 \leqslant s$ and $l_1' \leqslant s$. (Here, note that the ρ_l ($l = 0, 1,\ldots, s$) are irreducible orthogonal representations.) Now, in order to prove the assertion (i), we may assume that $\mathbf{X} = {}^t(1, 0,\ldots, 0)$ and we have only to show that

$$\sum_{g\in G} f(x_{11}(g), x_{21}(g),\ldots, x_{d1}(g)) = 0$$

for all homogeneous harmonic polynomials f of degree 1, 2,..., $2s$. Now, on the sphere Ω_d, any homogeneous polynomial f of degree l is expressed uniquely as

$$f = f_l + f_{l-2} + \cdots + f_{l-2[l/2]},$$

where f_i is a homogeneous harmonic polynomial of degree i. So, let f be any polynomial of degree at most $2s$. Then f is expressed as a certain linear combination of the $f_i f_j$ with $0 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant s$, where f_i denotes a homogeneous harmonic polynomial of degree i, because any polynomial of degree at most $2s$ is expressed as a linear combination of the products of two polynomials each of whose degree is $\leqslant s$. Of course, there are many ways to express the polynomial f in this way. However,

$$\int_{\Omega_d} f(x)\, d\Omega$$

does not depend on the way of expression. Now, from the formulas (4) and (5), we obtain that

$$\frac{1}{|G|} \sum_{g \in G} f(g) = \int_{\Omega_d} f(x)\, d\Omega,$$

for any polynomial f of degree at most $2s$.

Now, if f is any nonconstant homogeneous harmonic polynomial, then

$$\int_{\Omega_d} f(x)\, d\Omega = 0.$$

Thus, the assertion (i) is verified. We omit the proof of the assertion (ii), because it is easily obtained by slightly modifying the above argument that proved the assertion (i).

Concluding Remarks. (i) The above theorems give many examples of spherical t-designs, but for relatively small values of t. For example, the Weyl group of type E_8 satisfies the assumptions (i) and (ii) in Theorem 2 for $s = 3$ (cf. [4]). So, we get many examples of spherical 7-designs. Also, we will be able to obtain some other interesting examples by looking at some real reflection groups and various subgroups of Conway's groups (cf. [3].) However, it is not known and very unlikely that there is a finite group G for which the assumption in either (i) or (ii) in Theorem 2 is satisfied for large s (say for $s \geqslant 6$). So, as far as we are looking for spherical t-designs on which a group acts transitively, it seems very difficult (or might be unlikely) that we can find such a spherical t-design for large t. However, it would be very nice, if we could know whether there exists a spherical t-design with arbitrary large t for a fixed $d \geqslant 3$.

(ii) One may notice a strong analogy between Theorem 2 and the theorems which show the multiple transitivity of finite permutation groups by using the irreducibility of the restriction of certain characters of symmetric groups (cf., for example, [7].)

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